

# Moments of the cot function, central factorial numbers and their links with the Dirichlet eta function at odd integers

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## Abstract

We investigate the properties of the moments of the cot function using the central factorial numbers. Using a new integral representation of the central factorial numbers, we find a new way to express these moments in terms of recursive sums and integrals. This allows us to compute 'recursive' generalized harmonic series and multiple integrals as a linear combination of the Dirichlet eta functions at odd integers.

**Keywords**— cot function, central factorial numbers, Dirichlet eta function

## 1 Introduction

**The problem** Let  $C(m)$  denote the  $m$ th moment of the cot function, we define by the formula

$$C(m) \stackrel{\text{def}}{=} \frac{1}{m!} \int_0^\pi \frac{\theta^m}{2} \cot\left(\frac{\theta}{2}\right) d\theta, \quad m = 1, 2, \dots \quad (1)$$

There are few results about these moments. The reference text by Prudnikov et al. ([7], pp. 436) gives that

$$C(m) = \frac{\pi^n}{m!} \left[ \frac{1}{m} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(n+2k)} \right]$$

which we not find very useful and informative. Alternatively, using the Riemann-Lebesgue lemma, one has

$$C(m) = \frac{2}{m!} \sum_{n=1}^{\infty} \int_0^\pi \theta^m \sin(n\theta) d\theta.$$

The integrals involving  $\theta^m \sin(n\theta)$  can be evaluated for any pair  $(m, n) \in \mathbb{N}^2$  (see [2], 3.761-5). The resulting series give a linear combination of the Dirichlet eta function and, when  $m$  is even, the zeta function. More specifically, we have

$$C(m) = \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^l \frac{\pi^{m-2l}}{(m-2l)!} \eta(2l+1) + \frac{1+(-1)^m}{2} (-1)^{\lfloor \frac{m}{2} \rfloor} \zeta(m+1). \quad (2)$$

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The extra term appearing when  $m$  is even seems to indicate that there exists a different behavior of the moments of the function  $\cot$  for the odd and even moments. This is what will highlight the results presented hereafter (theorems 1.1, 1.2, 1.3).

Our approach in order to evaluate the moments of the  $\cot$  function is very different from the one used above: we first observe that by using an elementary change of variable, it is possible to make the generating function of the cfns appear inside the integral (1). Integrating this generating function term by term and applying a novel integral representation of the cfns (theorem 3.1), we arrive at a new series representation of these moments.

By comparing these two series, we can then evaluate a family of integrals and series that do not seem obvious to evaluate directly. Finally, we point out that (maybe) the most interesting results that may be found in this paper are the proof of the intermediary results. In fact, in Section 2, we evaluate exactly 'recursive' generalized harmonic series using a method that seems quite new, and we did not find elsewhere.

**Main results** Before stating our main results and some of their consequences, we recall an elementary identity that we will use throughout this paper. For any  $j > -1$  and  $l = 0, 1, \dots$ , we have

$$\frac{1}{(j+1)^{l+1}} = \frac{(-1)^l}{l!} \int_0^1 x^j \log^l(x) dx. \quad (3)$$

Following the convention used throughout this paper, we denote by  $C_0(k)$  the even moments of the  $\cot$  function (when  $m = 2k$ ,  $k > 0$ ) and by  $C_1(k)$  the odd moments of the  $\cot$  function (when  $m = 2k+1$ ,  $k \geq 0$ ). Our first result gives an integral representation for the moments of the  $\cot$  function in terms of two kernel functions

**Theorem 1.1.** For any  $k \geq 0$ , we have

$$C_1(k) = \sum_{l=0}^k 2^{2l+1} \frac{\pi^{2k-2l}}{(2k-2l)!} \int_{C_l} K_1(x_1 \dots x_l) \prod_{i=1}^l \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2}, \quad (4)$$

$$C_0(k+1) = \sum_{l=0}^k \frac{\pi^{2k-2l}}{(2k-2l+1)!} \int_{C_l} K_0(x_1 \dots x_l) \prod_{i=1}^l \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m} \quad (5)$$

with  $K_1$  and  $K_0$  given by the entire series

$$K_1(z) = \sum_{j=0}^{\infty} \binom{2j}{j} \frac{1}{(2j+1)^2} \left(\frac{z}{2}\right)^{2j} \quad \text{and} \quad K_0(z) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{(4z)^j}{j^3 \binom{2j}{j}},$$

$C_r$  denoting the unit square in  $\mathbb{R}^r$  and using the convention that the integrals are equal to one when  $r = 0$ .

It is not difficult to verify that the kernels  $K_1$  and  $K_0$  have an explicit form (see [4] for the second one) given by

$$K_1(z) = \frac{1}{z} \int_0^z \frac{\arcsin(y)}{y} dy \quad \text{and} \quad K_0(z) = 2 \int_0^z \frac{\arcsin(\sqrt{y})}{y} dy.$$

Based on this theorem, we can easily deduce the following theorem

**Theorem 1.2.** For any  $k \geq 0$ , we have

$$C_1(k) = - \sum_{l=0}^k 2^{2l+1} \frac{\pi^{2k-2l}}{(2k-2l)!} \int_{C_{l+1}} \frac{\log(x_0) dx_0}{\sqrt{1 - \prod_{i=0}^{k-l} x_i^2}} \prod_{i=1}^{k-l} \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} \quad (6)$$

$$C_0(k+1) = \sum_{l=0}^k \frac{\pi^{2k-2l}}{(2k-2l+1)!} \int_{C_{l+1}} \arcsin^2 \left( \sqrt{\prod_{i=0}^l x_i} \right) \frac{dx_0}{x_0} \prod_{i=1}^l \left( \frac{\log(x_i)}{1 - \prod_{m=1}^i x_m} \frac{dx_i}{x_i} \right) \quad (7)$$

*Proof.* We recall that for any  $|x| \leq 1$  we have (see [9] example 1.1.15 for the first identity and [4] for the second)

$$\sum_{j=0}^{\infty} \binom{2j}{j} \left(\frac{x}{2}\right)^{2j} = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{1}{2} \sum_{j=1}^{\infty} \frac{(2x)^{2j}}{j^2 \binom{2j}{j}} = \arcsin^2(x).$$

Using the identity (3) with  $j = 2$  and  $l = 1$ , we get  $1/(2j+1)^2 = - \int_0^1 x_0^{2j} \log(x_0) dx_0$  which gives us the first equality. For the second equality, write  $1/j = \int_0^1 x_0^{j-1} dx_0$  and conclude.  $\square$

Finally, we can also express theorem 1.1 as a sum of weighted recursive series

**Theorem 1.3.** For any  $k \geq 0$ , we have

$$C_1(k) = \sum_{l=0}^k (-1)^l \frac{2^{2l+1} \pi^{2k-2l}}{(2k-2l)!} S_1(l) \quad \text{and} \quad C_0(k+1) = \sum_{l=0}^k (-1)^l \frac{\pi^{2l}}{(2k-2l+1)!} S_0(l)$$

with

$$\begin{aligned} S_1(k) &= \sum_{j_0=0}^{\infty} \binom{2j_0}{j_0} \left(\frac{1}{2}\right)^{2j_0} \frac{1}{(2j_0+1)^2} \sum_{j_1=j_0}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_k=j_{k-1}}^{\infty} \frac{1}{(2j_1+1)^2 \cdots (2j_k+1)^2} \\ S_0(k) &= \frac{1}{2} \sum_{j_0=1}^{\infty} \frac{2^{2j_0}}{\binom{2j_0}{j_0}} \frac{1}{j_0^3} \sum_{j_1=j_0}^{\infty} \sum_{j_2=j_1}^{\infty} \cdots \sum_{j_k=j_{k-1}}^{\infty} \frac{1}{j_1^2 j_2^2 \cdots j_k^2} \end{aligned}$$

*Proof.* In equation 4, using a development in series of  $1/(1-x_1^2 \dots x_l^2)$ , we get

$$\begin{aligned} \int_{C_l} K_1(x_1 \dots x_l) \prod_{i=1}^l \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} &= \\ \int_{C_l} \sum_{j_0=0}^{\infty} \binom{2j_0}{j_0} \left(\frac{1}{2}\right)^{2j_0} \frac{1}{(2j_0+1)^2} \left( \sum_{j_1=0}^{\infty} (x_1 x_2 \dots x_l)^{2j_0+2j_1} \right) \log(x_l) dx_l &\prod_{i=1}^{l-1} \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} \end{aligned}$$

Integrating with respect to  $x_l$  (identity 3), we get

$$\begin{aligned} \int_{C_l} K_1(x_1 \dots x_l) \prod_{i=1}^l \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} &= \\ \int_{C_l} \sum_{j_0=0}^{\infty} \binom{2j_0}{j_0} \left(\frac{1}{2}\right)^{2j_0} \frac{1}{(2j_0+1)^2} \left( \sum_{j_1=j_0}^{\infty} \frac{-1}{(2j_1+1)^2} (x_1 x_2 \dots x_{l-1})^{2j_1} \prod_{i=1}^{l-1} \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} \right). \end{aligned}$$

Developing in series  $1/(1-x_1^2 \dots x_{l-1}^2)$  and integrating with respect to  $x_{l-1}$  and so on gives the announced result.

The proof of the second equality is similar and has been omitted.  $\square$

**Some consequences** Setting  $k=0$  and  $k=1$  and comparing the formula (2) with the representation given in Theorem 1.2 and 1.3, we obtain the following identities using the odd moments

$$\begin{aligned} \int_0^1 \frac{-\log(x_0)}{\sqrt{1-x_0^2}} dx_0 &= \sum_{j_0=0}^{\infty} \binom{2j_0}{j_0} \left(\frac{1}{2}\right)^{2j_0} \frac{1}{(2j_0+1)^2} = \frac{\pi}{2} \log(2), \\ \int_0^1 \int_0^1 \frac{\log(x_0) \log(x_1)}{\sqrt{1-x_0^2} \sqrt{1-x_1^2} (1-x_1^2)} dx_0 dx_1 &= \sum_{j_0=0}^{\infty} \binom{2j_0}{j_0} \left(\frac{1}{2}\right)^{2j_0} \frac{1}{(2j_0+1)^2} \sum_{j_1=j_0}^{\infty} \frac{1}{(2j_1+1)^2} = \frac{\pi^3}{24} \log(2) + \frac{\pi}{8} \eta(3). \end{aligned}$$

Using the even moments, we get

$$\begin{aligned} \int_0^1 \frac{\arcsin^2(\sqrt{x_0})}{x_0} dx_0 &= \frac{1}{2} \sum_{j_0=1}^{\infty} \frac{2^{2j_0}}{\binom{2j_0}{j_0}} \frac{1}{j_0^3} = \frac{\pi^2}{2} \log(2) - \frac{7}{3} \eta(3) \\ \int_0^1 \int_0^1 \frac{\arcsin^2(\sqrt{x_0 x_1})}{x_0 x_1} \frac{\log(x_1)}{1-x_1} dx_0 dx_1 &= \frac{-1}{2} \sum_{j_0=1}^{\infty} \frac{2^{2j_0}}{\binom{2j_0}{j_0}} \frac{1}{j_0^3} \sum_{j_1=j_0}^{\infty} \frac{1}{j_1^2} = -\frac{\pi^4}{24} \log(2) - \frac{\pi^2}{9} \eta(3) + \frac{31}{15} \eta(5). \end{aligned}$$

It is possible to find similar (and more complicated) expressions with  $k=2$ , etc., but it does not give a general formula for these integrals/recursive sums.

## 2 Evaluation of recursive series involving power of integers

In this section, we are proving some intermediate results that are interesting in their own right. In part 2.1, we evaluate exactly the recursive generalized harmonic series

$$R_1(k) \stackrel{\text{def}}{=} \sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \sum_{i_2=0}^{i_1} \frac{1}{(2i_2+1)^2} \cdots \sum_{i_k=0}^{i_{k-1}} \frac{1}{(2i_k+1)^2}$$

$$R_0(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{i_1^2} \sum_{i_2=1}^{i_1} \frac{1}{i_2^2} \cdots \sum_{i_k=1}^{i_{k-1}} \frac{1}{i_k^2}.$$

and in part 2.2 the “dual” series

$$A_1(k) \stackrel{\text{def}}{=} \sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \sum_{i_2=i_1+1}^{\infty} \frac{1}{(2i_2+1)^2} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} \frac{1}{(2i_k+1)^2} \quad (8)$$

$$A_0(k) \stackrel{\text{def}}{=} \sum_{i_1=1}^{\infty} \frac{1}{i_1^2} \sum_{i_2=i_1+1}^{\infty} \frac{1}{i_2^2} \cdots \sum_{i_k=i_{k-1}+1}^{\infty} \frac{1}{i_k^2}, \quad (9)$$

with the convention  $A_1(0) = A_0(0) = 1$ .

### 2.1 Evaluation of the series $R_1(k)$ and $R_0(k)$

In order to prove the next results, we use a result of (Vella, [10]) which is an application of the Faà di Bruno (1855) result. We believe that this usage is new.

**Theorem 2.1.** *For any  $k > 0$ , we have*

$$R_1(k) = \left(\frac{\pi}{2}\right)^{2k} \frac{E_{2k}^*}{(2k)!}$$

with  $E_{2k}^*$  denoting the Euler (zig) numbers ([OEIS A000364](#), even powers only of  $\sec(x) = 1/\cos(x)$ ).

*Proof.* Observe first that for any  $l \geq 1$ , we have

$$\sum_{i=0}^{\infty} \frac{1}{(2i+1)^{2l}} = \sum_{i=1}^{\infty} \frac{1}{i^{2l}} - \frac{1}{2^{2l}} \sum_{i=1}^{\infty} \frac{1}{i^{2l}} = \frac{2^{2l}-1}{2^{2l}} \zeta(2l) = (2^{2l}-1) \frac{|B_{2l}| \pi^{2l}}{2(2l)!}, \quad (10)$$

with  $B_n$  denoting the Bernoulli numbers. For  $n \geq 1$ , let  $x_n = 1/(2n-1)^2$  and fix  $0 < k \leq n$ . The complete homogeneous symmetric polynomials of degree  $k$  in  $x_1, \dots, x_n$  is defined as

$$h_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Let  $\mathcal{P}(k)$  denote the set of all partitions of the integer  $k$ . For  $\pi \in \mathcal{P}(k)$  we write  $\pi = [1^{\pi_1}, \dots, k^{\pi_k}]$  with  $\pi_l$  denoting the multiplicities of the part  $l$  in the current partition of  $k$ . With this convention, we have trivially, for any  $\pi \in \mathcal{P}(k)$ ,  $k = \sum_{l=1}^k l\pi_l$  and we define the length of the partition as  $l(\pi) = \sum_{l=1}^k \pi_l$ . Using the cycle index formula ([12]), we have

$$k! h_k(x_1, \dots, x_n) = \sum_{\pi \in \mathcal{P}(k)} a(\pi) \prod_{l=1}^k \left( \sum_{j=1}^n x_j^l \right)^{\pi_l}$$

with  $a(\pi)$  denoting the number of permutations whose cycle structure is the given partition. These numbers  $a(\pi)$  have the explicit expression

$$a(\pi) = k! \prod_{l=1}^k \frac{1}{\pi_l! l^{\pi_l}}, \quad \pi \in \mathcal{P}(k).$$

Taking the limit in  $n$ , and using (10), we get

$$R_1(k) = \frac{1}{k!} \sum_{\pi \in \mathcal{P}(k)} a(\pi) \prod_{l=1}^k \left[ \frac{2^{2l} - 1}{2^{2l}} \zeta(2l) \right]^{\pi_l}.$$

Simplifying, we obtain the following

$$R_1(k) = \pi^{2k} \sum_{\pi \in \mathcal{P}(k)} \prod_{l=1}^k \frac{1}{\pi_l!} \left[ (2^{2l} - 1) \frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l} = \pi^{2k} \sum_{\pi \in \mathcal{P}(k)} \binom{l(\pi)}{\pi_1, \dots, \pi_k} \frac{1}{l(\pi)!} \prod_{l=1}^k \left[ (2^{2l} - 1) \frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l},$$

with  $\binom{n}{\alpha_1, \dots, \alpha_k}$  denoting the multinomial coefficient. Using the generating function of the Bernoulli numbers it is easily found that

$$\frac{z}{2} \cot\left(\frac{z}{2}\right) - z \cot(z) = \sum_{n=1}^{\infty} (2^{2n} - 1) |B_{2n}| \frac{z^{2n}}{(2n)!}.$$

Consider the entire series

$$g(z) = \sum_{n=1}^{\infty} (2^{2n} - 1) |B_{2n}| \frac{z^{2n}}{(2n)(2n)!}.$$

Derivating with respect to  $z$ , we get

$$zg'(z) = \frac{z}{2} \cot\left(\frac{z}{2}\right) - z \cot(z)$$

and thus

$$g(z) = \log\left(\frac{\sin(z/2)}{\sin(z)}\right) + \log(2) = \log\left(\frac{1}{\cos(z/2)}\right)$$

is the generating function of the sequence  $(2^{2n} - 1) \frac{|B_{2n}|}{(2n)(2n)!}$ . We apply Corollary 4 of (Vella, [10]) with  $g$  given above and  $f(z) = \exp(z)$ . Recognizing that  $1/\cos(z/2)$  is the generating function of  $E_{2n}^*/2^n$ , we arrive at the conclusion.  $\square$

We have a similar theorem for the series  $R_0(k)$ .

**Theorem 2.2.** *For any  $k > 0$ , we have*

$$R_0(k) = 2(2^{2k-1} - 1) \frac{|B_{2k}| \pi^{2k}}{(2k)!}.$$

*Proof.* The arguments are very similar to the previous one, so we give only a sketch of the proof. Using the same methodology, we arrive at the expression

$$R_0(k) = \frac{1}{k!} \sum_{\pi \in \mathcal{P}(k)} a(\pi) \prod_{l=1}^k \zeta(2l)^{\pi_l}.$$

Simplifying, we get the following

$$R_0(k) = (2\pi)^{2k} \sum_{\pi \in \mathcal{P}(k)} \prod_{l=1}^k \frac{1}{\pi_l!} \left( \frac{|B_{2l}|}{2l(2l)!} \right)^{\pi_l} = (2\pi)^{2k} \sum_{\pi \in \mathcal{P}(k)} \binom{l(\pi)}{\pi_1, \dots, \pi_k} \frac{1}{l(\pi)!} \prod_{l=1}^k \left[ \frac{|B_{2l}|}{2l(2l)!} \right]^{\pi_l}.$$

Next, we find that

$$\log(z/2) - \log(\sin(z/2)) = \sum_{n=1}^{\infty} \frac{|B_{2n}|}{2n} \frac{z^{2n}}{(2n)!}$$

and that  $g(z) = \frac{z}{2} \frac{1}{\sin(z/2)}$  is the generating function of the sequence

$$2(2^{2n-1} - 1) \frac{|B_{2n}|}{(2n)! 2^{2n}}, \quad n \geq 1.$$

See also [OEIS A036280](#) and [OEIS A036281](#).  $\square$

## 2.2 Evaluation of the series $A_1(k)$ and $A_0(k)$

From the previous results, we obtain the corollary

**Corollary 2.3.** *For any  $k > 0$ , the following holds*

$$A_1(k) = \left(\frac{\pi}{2}\right)^{2k} \frac{1}{(2k)!} \quad \text{and} \quad A_0(k) = \frac{\pi^{2k}}{(2k+1)!}.$$

*Proof.* We just need to prove the first claim as the evaluation of  $A_0(k)$  is already proved by [3] and [5] using similar arguments to the one used below.

It is easy to find that when  $k = 1$ ,  $A_1 = \pi^2/8$  but the remaining constants are more difficult to compute. Let thus consider  $k \geq 2$ , with an obvious meaning for the coefficient  $a_i^{(k)}$ , we have

$$\begin{aligned} A_1(k) &= \sum_{i_1=0}^{\infty} \frac{a_{i_1}^{(k)}}{(2i_1+1)^2} = \sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \left( A_{k-1} - \sum_{i_2=0}^{i_1} \frac{a_{i_2}^{(k-1)}}{(2i_2+1)^2} \right) \\ &= R_1(1)A_1(k-1) - \left[ \sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \sum_{i_2=1}^{i_1} \frac{a_{i_2}^{(k-1)}}{(2i_2+1)^2} \right] \\ &= \dots \\ &= \sum_{l=1}^k (-1)^{l+1} R_1(l) A_1(k-l). \end{aligned}$$

with  $R_1(l)$  defined in theorem 2.2. The result is proven by recurrence. Assuming the expression of  $A_1(l)$  is true for  $1 \leq l \leq k-1$ , it will be true for  $l = k$  if

$$\sum_{l=1}^k (-1)^{l+1} \binom{2k}{2l} E_{2l}^* = 1 \quad \Longleftrightarrow \quad \sum_{l=0}^k (-1)^{l+1} \binom{2k}{2l} E_{2l}^* = 0.$$

This result can be found in [8] and on the web site [proofwiki](#) (Sum of Euler Numbers by Binomial Coefficients Vanishes). □

## 2.3 Integral representation of $A_1(k)$ and $A_0(k)$

The next lemma gives an integral representation of the series  $A_0(k)$  and  $A_1(k)$

**Lemma 2.4.** *For any  $k \geq 0$ , the following equality holds*

$$A_1(k) = (-1)^k \int_{C_k} \prod_{i=1}^k \frac{x_i^{2(i-1)} \log(x_i) dx_i}{\left(1 - \prod_{m=i}^k x_m^2\right)} \quad \text{and} \quad A_0(k) = (-1)^k \int_{C_k} \prod_{i=1}^k \frac{x_i^{i-1} \log(x_i) dx_i}{\left(1 - \prod_{m=i}^k x_m\right)}$$

with  $A_0(k)$  and  $A_1(k)$  defined in (9) and (8).

*Proof.* We prove the result for  $A_1(k)$ . The proof for  $A_0(k)$  is similar and is omitted. We have to show that the integral on the right hand side is equal to

$$\sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \sum_{i_2=i_1+1}^{\infty} \frac{1}{(2i_1+1)^2} \cdots \sum_{i_l=i_{l-1}+1}^{\infty} \frac{1}{(2i_k+1)^2}.$$

Developing in series the (unique) term with  $x_1^2$  we obtain

$$\frac{1}{1 - x_1^2 x_2^2 \dots x_k^2} = \sum_{i_1=0}^{\infty} (x_1^2 x_2^2 \dots x_k^2)^{i_1}$$

and integrating with respect to  $\log(x_1) dx_1$  we get that the right hand side is equal to

$$\sum_{i_1=0}^{\infty} \frac{1}{(2i_1+1)^2} \int_{C_{k-1}} \frac{(-1)^{k-1} \prod_{i=2}^k x_i^{2(i-1)+2i_1} \log(x_i) dx_i}{\prod_{i=2}^l \left(1 - \prod_{m=i}^k x_m^2\right)}.$$

Iterating the process with  $x_2, x_3, \dots, x_k$  we arrive at the conclusion. □

### 3 Integral representation of the central factorial numbers (cfns)

In this part, we define the (unsigned) central factorial numbers and give an integral representation of the "multinomial" part. The sequences  $t_0(k, n) = |t(2n, 2k)|$  ([OEIS A008955](#)) and  $t_1(k, n) = |t(2n+1, 2k+1)|$  ([OEIS A008956](#)) used in this paper are obtained from the recurrences

$$t_0(k, n) = t_0(k-1, n-1) + (n-1)^2 t_0(k, n-1), \quad t_1(k, n) = t_1(k-1, n-1) + \left(n - \frac{1}{2}\right)^2 t_1(k, n-1)$$

with diagonal and lower part given by

$$t_0(n, n) = t_1(n, n) = 1, \text{ for all } n \in \mathbb{N}, \quad t_0(k, n) = t_1(k, n) = 0, \text{ if } k < n,$$

and row of index zero given by

$$t_0(0, n) = 0, \quad t_1(0, n) = \frac{(2n+1)!!^2}{4^n}, \quad \text{if } n > 0.$$

The first cfns for  $0 \leq k \leq 5$  and  $0 \leq n \leq 5$  are given in table 1.

$k \backslash n$	0	1	2	3	4	5	...
0	1	0	0	0	0	0	...
1	0	1	1	4	36	576	...
2	0	0	1	5	49	820	...
3	0	0	0	1	14	273	...
4	0	0	0	0	1	30	...
5	0	0	0	0	0	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$k \backslash n$	0	1	2	3	4	5	...
0	1	$\frac{1}{4}$	$\frac{9}{16}$	$\frac{225}{64}$	$\frac{11025}{3229}$	$\frac{893025}{1057221}$	...
1	0	1	$\frac{5}{2}$	$\frac{259}{16}$	$\frac{3229}{8}$	$\frac{1057221}{86405}$	...
2	0	0	1	$\frac{35}{4}$	$\frac{987}{8}$	$\frac{256}{4389}$	...
3	0	0	0	1	21	$\frac{8}{165}$	...
4	0	0	0	0	1	$\frac{1}{4}$	...
5	0	0	0	0	0	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1: Arrays of the central factorial number. Left: the numbers  $t_0$ . Right: the numbers  $t_1$

These sequences are related to the central factorial numbers (cfn)  $t(2n, 2k)$  and  $t(2n+1, 2k+1)$  presented and studied in (Butzer and al. [1]). Note that, with the convention used in this paper, cfns  $t_0$  and  $t_1$  are transposed and unsigned with respect to the sequences studied in the cited article. Using the previous reference, we have also that the cfns  $|t(n, m)|$  have the following generating function ([1], Theorem 4.1.2) for  $|z| < 2$  and any  $m \in \mathbb{N}$

$$\frac{1}{m!} \left[ 2 \arcsin \left( \frac{z}{2} \right) \right]^m = \sum_{n=m}^{\infty} |t(n, m)| \frac{z^n}{n!}. \quad (11)$$

In particular, if  $m = 2k$  and  $m = 2k+1$ , we have

$$\frac{1}{(2k)!} \left[ 2 \arcsin \left( \frac{z}{2} \right) \right]^{2k} = \sum_{n=k}^{\infty} t_0(k, n) \frac{z^{2n}}{(2n)!}, \quad k \geq 1 \quad (12)$$

$$\frac{1}{(2k+1)!} \left[ 2 \arcsin \left( \frac{z}{2} \right) \right]^{2k+1} = \sum_{n=k}^{\infty} t_1(k, n) \frac{z^{2n+1}}{(2n+1)!}, \quad k \geq 0. \quad (13)$$

Finally, we define the following recursive harmonic numbers of order two as  $H_0(k, n)$  and  $H_1(k, n)$  by the initial conditions  $H_0(0, 0) = 1$ ,  $H_0(0, n) = 0$  if  $n \geq 1$ ,  $H_0(1, n) = H_1(0, n) = 1$  for all  $n > 0$ , and, if  $k > 1$ , the recurrences

$$H_0(k, n) = \sum_{i_k=k-1}^{n-1} \frac{1}{i_k^2} H_0(k-1, i_k) = \sum_{i_k=k-1}^{n-1} \frac{1}{i_k^2} \left( \sum_{i_{k-1}=k-2}^{i_k-1} \frac{1}{i_{k-1}^2} \cdots \sum_{i_1=1}^{i_2-1} \frac{1}{i_1^2} \right)$$

$$H_1(k, n) = \sum_{i_k=k-1}^{n-1} \frac{1}{(2i_k+1)^2} H_1(k-1, i_k) = \sum_{i_k=k-1}^{n-1} \frac{1}{(2i_k+1)^2} \left( \sum_{i_{k-1}=k-2}^{i_k-1} \frac{1}{(2i_{k-1}+1)^2} \cdots \sum_{i_1=0}^{i_2-1} \frac{1}{(2i_1+1)^2} \right).$$

$k \backslash n$	0	1	2	3	4	5	...
0	1	0	0	0	0	0	...
1	0	1	1	1	1	1	...
2	0	0	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	...
3	0	0	0	$\frac{1}{4}$	$\frac{7}{18}$	$\frac{91}{192}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$k \backslash n$	0	1	2	3	4	5	...
0	1	1	1	1	1	1	...
1	0	1	$\frac{10}{9}$	$\frac{259}{225}$	$\frac{12916}{11025}$	$\frac{117469}{99225}$	...
2	0	0	$\frac{1}{9}$	$\frac{45}{7}$	$\frac{525}{4}$	$\frac{94}{178605}$	...
3	0	0	0	$\frac{1}{225}$	$\frac{1}{525}$	$\frac{418}{42525}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2: Arrays of the recursive harmonic numbers of order two. Left: the numbers  $H_0$ . Right: the numbers  $H_1$

The first numbers for  $0 \leq k \leq 3$  and  $0 \leq n \leq 5$  are given in table 2. Finally, for all  $k \in \mathbb{N}$  ([1], p. 430)

$$t_0(k, n) = (n-1)!^2 H_0(k, n), \quad n \geq 1, \quad \text{and} \quad t_1(k, n) = 2^{2k} \binom{2n}{n} \frac{(2n)!}{2^{4n}} H_1(k, n), \quad n \geq 0.$$

The next theorem gives an integral representation of the coefficients  $H_1(k, j)$  and  $H_0(k, j)$ .

**Theorem 3.1.** *Let  $k \geq 0$ , for any  $j \geq 0$ , the coefficients  $H_0(k+1, j)$  and  $H_1(k, j)$  can be expressed as the sum of integrals of the form*

$$H_1(k, j) = \sum_{l=0}^k \frac{1}{(2l)!} \left(\frac{\pi}{2}\right)^{2l} \int_{C_{k-l}} \prod_{i=l+1}^k \frac{x_i^{2j} \log(x_i) dx_i}{\left(1 - \prod_{m=l+1}^i x_m^2\right)} \quad (14)$$

$$H_0(k+1, j) = \sum_{l=0}^k \frac{\pi^{2l}}{(2l+1)!} \int_{C_{k-l}} \prod_{i=l+1}^k \frac{x_i^{j-1} \log(x_i) dx_i}{\left(1 - \prod_{m=l+1}^i x_m\right)} \quad (15)$$

for  $l = 0, \dots, k$ , with  $C_l$  denoting the unit cube in  $\mathbb{R}^l$  and with the convention that the integrals in the right-hand sides are equal to 1 when  $l = k$ .

*Proof.* We first show that for  $j \geq k$ , the coefficients  $H_1(k, j)$  can be expressed as integral of the form

$$H_1(k, j) = (-1)^k \int_0^1 \dots \int_0^1 \sum_{l=0}^k r_l^{(k)}(x_1, \dots, x_k) x_{l+1}^{2j} \dots x_k^{2j} \log(x_1) \dots \log(x_k) dx_1 \dots dx_k \quad (16)$$

with

$$r_l^{(k)}(x_1, \dots, x_k) = \frac{(-1)^{k+l} \prod_{i=1}^l x_i^{2(i-1)}}{\prod_{i=1}^l \left(1 - \prod_{m=i}^l x_m^2\right) \prod_{i=l+1}^k \left(1 - \prod_{m=l+1}^i x_m^2\right)}, \quad (17)$$

independent of  $j$ , for  $l = 0, \dots, k$ . Using repeatedly the identity (3) with  $l = 1$ , we obtain

$$H_1(k, j) = (-1)^k \int_0^1 \dots \int_0^1 \left( \sum_{i_k=k-1}^{j-1} x_k^2 \left( \sum_{i_{k-1}=k-2}^{i_k-1} x_{k-1}^2 \dots \sum_{i_1=0}^{i_2-1} x_1^2 \right) \right) \prod_{j=1}^k \log(x_j) dx_j.$$

If  $|z| < 1$ , then

$$\sum_{i_l=l-1}^{i_{l+1}-1} z^{2i_l} = \frac{z^{2l-2} - z^{2i_{l+1}}}{1 - z^2}, \quad l = 1, \dots, k.$$

Using this identity, we obtain that the sum inside the integral (replacing  $j$  by  $i_{k+1}$ ) has the expression

$$\left( \sum_{i_k=k-1}^{i_{k+1}-1} x_k^2 \dots \sum_{i_1=0}^{i_2-1} x_1^2 \right) = \sum_{l=0}^k r_l^{(k)}(x_1, \dots, x_k) \prod_{m=l+1}^k x_m^{2i_{k+1}}$$



and that  $r_l^{(k)}(x_1, \dots, x_k)$  obey the following recurrence relations in  $k$

$$\begin{aligned} r_l^{(k)}(x_1, \dots, x_k) &= r_l^{(k-1)}(x_1, \dots, x_{k-1}) \frac{-1}{1 - (x_{l+1} \dots x_k)^2}, \quad \text{if } 0 \leq l < k, \\ r_k^{(k)}(x_1, \dots, x_k) &= \sum_{j=0}^{k-1} r_j^{(k-1)}(x_1, \dots, x_{k-1}) \frac{(x_{j+1} \dots x_k)^{2k}}{1 - (x_j \dots x_k)^2}, \quad \text{otherwise.} \end{aligned}$$

with initial value, when  $k = 1$ ,

$$r_0^{(1)}(x_1) = \frac{-1}{1 - x_1^2}, \quad r_1^{(1)}(x_1) = \frac{x_1^0}{1 - x_1^2}.$$

This show that if  $0 \leq l < k$ , we have

$$r_l^{(k)}(x_1, \dots, x_k) = r_l^{(l)}(x_1, \dots, x_l) \frac{(-1)^{k-l}}{(1 - x_{l+1}^2) \dots (1 - (x_{l+1} \dots x_k)^2)}$$

It remains to verify that  $r_l^{(l)}(x_1, \dots, x_l)$  is given by

$$r_l^{(l)}(x_1, \dots, x_l) = \frac{x_1^0 x_2^2 \dots x_l^{2l-2}}{(1 - x_1^2) \dots (1 - x_1^2 \dots x_l^2)} \quad (18)$$

for  $1 \leq l \leq k$ . This can be done by recurrence. Let us assume that equation (18) is fulfilled for  $j = 1, \dots, k-1$ . Then  $r_k^{(k)}$  verify the equation

$$r_k^{(k)}(x_1, \dots, x_k) = \sum_{j=0}^{k-1} (-1)^{k-j} \frac{r_j^{(j)}(x_1, \dots, x_j)}{(1 - x_{j+1}^2) \dots (1 - (x_{j+1} \dots x_k)^2)},$$

using the recurrence assumption and summing term by term, we obtain the claim.

Thus, the expression given in equation (16) can be written

$$H_1(k, j) = \sum_{l=0}^k \int_{C_l} \prod_{i=1}^l (-1)^i \frac{x_i^{2(i-1)} \log(x_i) dx_i}{(1 - \prod_{m=i}^l x_m^2)} \times \int_{C_{k-l}} \prod_{i=l+1}^k \frac{x_i^{2j} \log(x_i) dx_i}{(1 - \prod_{m=l+1}^i x_m^2)}$$

with the convention that when  $l = 0$ , the integral is equal to one. The first integral does not depend on  $j$  and is equal to  $A_1(l)$  thanks to the lemma 2.4.

Finally, we observe that the polynomials  $r_l^{(k)}(x_1, \dots, x_k) x_{l+1}^{2j} \dots x_k^{2j}$  are equal to zero if  $j < k$ , thus the enunciated result is true for any  $j \geq 0$ . This ends the claim for  $H_1$  and the proof of the first part of the theorem.

The proof for  $H_0(k, j)$  is similar (just replace  $x_i^2$  by  $x_i$ ) and is omitted.  $\square$

## 4 Proof of the main results

Making the change of variable  $v = 2 \sin(\theta/2)$ ,  $\partial v = \cos(\theta/2) \partial \theta$  in the integral (1), we obtain

$$C(m) = \int_0^2 \frac{1}{m!} \left[ 2 \arcsin \left( \frac{v}{2} \right) \right]^m \frac{dv}{v}.$$

We recognize in this integral the generating function of the (unsigned) central factorial numbers (equation 11). Integrating term by term, we get the following representation of the moment of the cot function in terms of the cfns

$$C(m) = \sum_{n=m}^{\infty} |t(n, m)| \frac{2^n}{n n!} \quad (19)$$

We get after simplification the following series representations for the odd and even moments of the cot function

$$\begin{aligned} C_1(k) &= 2^{2k+1} \sum_{j=0}^{\infty} \binom{2j}{j} \frac{1}{4^j} \frac{H_1(k, j)}{(2j+1)^2} \\ C_0(k) &= \frac{1}{2} \sum_{j=1}^{\infty} \frac{2^{2j}}{j^3 \binom{2j}{j}} H_0(k, j). \end{aligned}$$

We replace  $H_1(k, j)$  and  $H_0(k, j)$  in the previous expressions using Theorem 3.1. Inverting sum and integral, we arrive to the identities

$$\begin{aligned} C_1(k) &= 2^{2k+1} \sum_{l=0}^k \frac{1}{(2l)!} \left(\frac{\pi}{2}\right)^{2l} \int_{C_{k-l}} K_1(x_1 \dots x_{k-l}) \prod_{i=1}^{k-l} \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m^2} \\ C_0(k) &= \sum_{l=0}^k \frac{\pi^{2l}}{(2l+1)!} \int_{C_{k-l}} K_0(x_1 \dots x_{k-l}) \prod_{i=1}^{k-l} \frac{\log(x_i) dx_i}{1 - \prod_{m=1}^i x_m}. \end{aligned}$$

Summing in a reversed order gives the result.

## 5 Numerical validations

All presented results have been thoroughly checked using the Python symbolic calculus library `sympy` ([6]) and/or the Python numerical calculus library `scipy` ([11]). The code (in the form of Jupyter notebooks) is available on demand.

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