

New proof of the Gaussian integral using the residue theorem with links to the Riemann Zeta function

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Abstract

In this paper the Gaussian integral is proven using contour integration on $\frac{1}{e^{z^2}+1}$ and linking it using a limit to said Gaussian integral. The limit is also related to the Riemann Zeta function using a few manipulations. This new and original proof comes as an addition to the already many pre-existing proofs of the Gaussian integral [1]

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1 The proof

Let us find $\int_{-\infty}^{\infty} \frac{1}{e^{x^2}+1} dx$, because z^2 and e^z are holomorphic, and $1/(z+1)$ is also holomorphic, we get that $1/(e^{z^2}+1)$ is holomorphic, hence we can use contour integration for our integral.

The poles of $1/(e^{z^2}+1)$ at $z = c$ correspond to the values of c such that $e^{c^2}+1=0$, $e^{c^2}=-1$:

$$e^{c^2} = e^{(2k+1)\pi i} \quad (k \in \mathbb{N})$$

$$c^2 = (2k+1)\pi i = (2k+1)\pi e^{i\pi/2}$$

With k running from $-\infty$ to ∞ , $2k+1$ can be reduced to $\pm(2k+1)$ with k running from 0 to ∞

$$c^2 = \pm(2k+1)\pi e^{i\pi/2} \quad (k \in \mathbb{N}, 0 \leq k)$$

$$c = \pm\sqrt{\pm\sqrt{(2k+1)\pi}e^{i\pi/4}} = \pm\sqrt{\pm\sqrt{(2k+1)\pi}}\frac{1+i}{\sqrt{2}} = \sqrt{(2k+1)\pi}\frac{\pm 1 + \pm i}{\sqrt{2}}$$

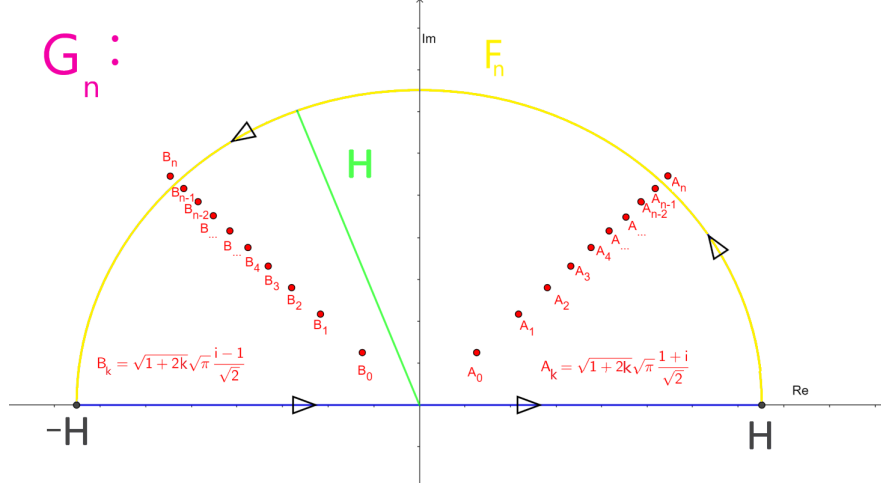
Our contour is going to be in the upper complex plane, hence poles with negative imaginary parts are irrelevant, all our relevant poles are then :

$$A_k = (i+1)\sqrt{\frac{(2k+1)\pi}{2}} \quad (0 \leq k, k \in \mathbb{N}) \tag{1}$$

$$B_k = (i-1)\sqrt{\frac{(2k+1)\pi}{2}} \quad (0 \leq k, k \in \mathbb{N})$$

H_n is a real value dependant of n . We take our contour, G_n , to be a line starting from $-H_n$ to H_n , then, a semi circle with positive imaginary value centered at 0 of radius H_n starting at the point H_n and ending back up at $-H_n$, called F_n . By taking H_n to be between the absolute value of A_n and A_{n-1} or B_n and B_{n-1} , this ensures we are not treading on any poles and that all the poles in our contour are A_k and B_k with k going from 0 to $n-1$. Integrating over G_n gives:

Figure 1: The contour G in the complex plane



$$\oint_{G_n} \frac{1}{e^{z^2} + 1} dz = \int_{-H_n}^{H_n} \frac{1}{e^{x^2} + 1} dx + \int_{F_n} \frac{1}{e^{z^2} + 1} dz$$

From our contour, F_n can be parametrized as :

$$F_n : z = H_n e^{iv} \quad 0 \rightarrow v \rightarrow \pi \quad dz = H_n i e^{iv} dv$$

$$\boxed{\oint_{G_n} \frac{1}{e^{z^2} + 1} dz = \int_{-H_n}^{H_n} \frac{1}{e^{x^2} + 1} dx + \int_0^\pi \frac{H_n i e^{iv}}{e^{H_n^2 e^{2iv}} + 1} dv} \quad (2)$$

Because the poles of $1/(e^{z^2} + 1)$ are of first order, the residue becomes :

$$\text{Res}_{z=c} \frac{1}{e^{z^2} + 1} = \lim_{z \rightarrow c} \frac{z - c}{e^{z^2} + 1}$$

Since both sides of the fraction go to zero, $\lim_{z \rightarrow c} \frac{1}{2ze^{z^2}}$ exists (c is nonzero) and because $\frac{1}{e^{z^2} + 1}$ is holomorphic, we can use L'Hopital's rule, such as :

$$\text{Res}_{z=c} \frac{1}{e^{z^2} + 1} = \lim_{z \rightarrow c} \frac{\frac{d(z-c)}{dz}}{\frac{d(e^{z^2} + 1)}{dz}} = \lim_{z \rightarrow c} \frac{1}{2ze^{z^2}} = \frac{1}{2ce^{c^2}} = \frac{-1}{2c}$$

H_n is between the absolute value of A_n and A_{n-1}

$$|A_{n-1}| = \sqrt{(2n-1)\pi} < H_n < \sqrt{(2n+1)\pi} = |A_n|$$

At some point, $k = n - 1$, because $0 \leq k, k \in \mathbb{N}$ this makes $1 \leq n, n \in \mathbb{N}$. From the residue theorem, because we have only A_k and B_k poles with k from 0 to $n - 1$ in our contour, from 2, 1 :

$$\begin{aligned}
\oint_{G_n} \frac{1}{e^{z^2} + 1} dz &= 2\pi i \sum_{k=0}^{n-1} \left(\operatorname{Res}_{z=A_k} \frac{1}{e^{z^2} + 1} + \operatorname{Res}_{z=B_k} \frac{1}{e^{z^2} + 1} \right) = 2\pi i \sum_{k=0}^{n-1} \left(\frac{-1}{2A_k} - \frac{1}{2B_k} \right) \\
&= 2\pi i \sum_{k=0}^{n-1} \left(\frac{-1}{2(i+1)\sqrt{\frac{(2k+1)\pi}{2}}} - \frac{1}{2(i-1)\sqrt{\frac{(2k+1)\pi}{2}}} \right) = \sum_{k=0}^{n-1} \frac{-\sqrt{2\pi}}{\sqrt{2k+1}} \\
&\quad \boxed{\int_{-H_n}^{H_n} \frac{dx}{e^{x^2} + 1} + \int_0^\pi \frac{H_n i e^{iv} dv}{e^{H_n^2 e^{2iv}} + 1} = \sum_{k=0}^{n-1} \frac{-\sqrt{2\pi}}{\sqrt{2k+1}}} \quad (3) \\
&\quad \boxed{\sqrt{(2n-1)\pi} < H_n < \sqrt{(2n+1)\pi}, \quad 1 \leq n, n \in \mathbb{N}}
\end{aligned}$$

Setting $\sqrt{(2n-1)\pi} < H_n = \sqrt{2n\pi} < \sqrt{(2n+1)\pi}$:

$$\boxed{\int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{1}{e^{x^2} + 1} dx + \sum_{k=0}^{n-1} \frac{\sqrt{2\pi}}{\sqrt{2k+1}} = - \int_0^\pi \frac{i e^{iv} \sqrt{2n\pi}}{e^{2n\pi e^{2iv}} + 1} dv \quad (1 \leq n, n \in \mathbb{N})} \quad (4)$$

Applying the real part to the equation does not change it because $\int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{e^{x^2} + 1}$ and $\sum_{k=0}^{n-1} \frac{\sqrt{2\pi}}{\sqrt{2k+1}}$ are real the equation is all real. Because Re commutes with integration, and that $\operatorname{Re}(iz) = -\operatorname{Im}(z)$:

$$- \int_0^\pi \frac{i e^{iv} \sqrt{2n\pi}}{e^{2n\pi e^{2iv}} + 1} dv = -\operatorname{Re} \left(\int_0^\pi \frac{i e^{iv} \sqrt{2n\pi}}{e^{2n\pi e^{2iv}} + 1} dv \right) = - \int_0^\pi \operatorname{Re} \left(\frac{i e^{iv} \sqrt{2n\pi}}{e^{2n\pi e^{2iv}} + 1} \right) dv$$

We then have, with defining $P(n)$

$$\boxed{P(n) = \int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{e^{x^2} + 1} + \sum_{k=0}^{n-1} \frac{\sqrt{2\pi}}{\sqrt{2k+1}} = \sqrt{n} \int_0^\pi \operatorname{Im} \left(\frac{e^{iv} \sqrt{2\pi}}{e^{2n\pi e^{2iv}} + 1} \right) dv \quad (1 \leq n, n \in \mathbb{N})} \quad (5)$$

Because the function $\frac{1}{e^{x^2} + 1}$ is positive, the greater our interval of integration is, the greater the integral of the function is, as n strictly increases, the interval from $-\sqrt{2n\pi}$ to $\sqrt{2n\pi}$ strictly increases, hence $P(n) = \int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{e^{x^2} + 1}$ is a strictly increasing and positive function.

From $1 + x \leq e^x$ we can have : $\frac{1}{e^{x^2} + 1} \leq \frac{1}{x^2 + 2}$

$$0 \leq P(n) = \int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{e^{x^2} + 2} \leq \int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{x^2 + 2} = \int_{-\sqrt{n\pi}}^{\sqrt{n\pi}} \frac{1/\sqrt{2}}{x^2 + 1} dx = \frac{2 \arctan(\sqrt{n\pi})}{\sqrt{2}} \leq \frac{\pi}{\sqrt{2}} \quad (6)$$

This makes $P(n)$ bounded between 0 and $\pi/\sqrt{2}$, from the monotone convergence theorem [2], if a sequence is increasing and bounded, it's limit is the supremum, applying it to $P(n)$ 5, remembering that n is strictly an integer :

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\sqrt{2n\pi}}^{\sqrt{2n\pi}} \frac{dx}{e^{x^2} + 1} &= \int_{-\infty}^{\infty} \frac{dx}{e^{x^2} + 1} = \limsup_{n \rightarrow \infty} \left(\sqrt{n} \int_0^{\pi} \operatorname{Im} \left(\frac{\sqrt{2\pi} e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv - \sum_{k=0}^{n-1} \frac{\sqrt{2\pi}}{\sqrt{2k+1}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\sqrt{n} \int_0^{\pi} \operatorname{Im} \left(\frac{e^{iv} \sqrt{2\pi}}{e^{2n\pi e^{2iv}} + 1} \right) dv - \sum_{k=0}^{n-1} \frac{\sqrt{2\pi}}{\sqrt{2k+1}} \right) \end{aligned}$$

Although the suprema can not be directly applied to $\sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}}$ since it does not converge alone, it can be applied by moving the limit of n to ∞ inside the integral such as :

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx}{e^{x^2} + 1} = \lim_{n \rightarrow \infty} \left(\sqrt{2n} \int_0^{\pi} \lim_{n \rightarrow \infty} \operatorname{Im} \left(\frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} \right) \quad (7)$$

Because limits and imaginary part commute :

$$\begin{aligned} \int_0^{\pi} \lim_{n \rightarrow \infty} \operatorname{Im} \left(\frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv &= \int_0^{\pi} \operatorname{Im} \left(\lim_{n \rightarrow \infty} \frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv \\ &= \int_0^{\pi} \operatorname{Im} \left(\lim_{n \rightarrow \infty} \frac{e^{iv}}{e^{2n\pi \cos(2v)} e^{2in\pi \sin(2v)} + 1} \right) dv \end{aligned}$$

For $0 \leq v < \pi/4$ and $3\pi/4 < v \leq \pi$, we have that $0 < \cos(2v)$. Because $|e^{2in\pi \sin(2v)}| = 1$, no matter what value $\lim_{n \rightarrow \infty} e^{2in\pi \sin(2v)}$ goes to, the integrated function goes to zero on that interval because $\lim_{n \rightarrow \infty} e^{2n\pi \cos(2v)}$ goes to infinity.

$$\int_0^{\pi} \lim_{n \rightarrow \infty} \operatorname{Im} \left(\frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv = \int_{\pi/4}^{3\pi/4} \operatorname{Im} \left(\lim_{n \rightarrow \infty} \frac{e^{iv}}{e^{2n\pi \cos(2v)} e^{2in\pi \sin(2v)} + 1} \right) dv$$

For $\pi/4 < v < 3\pi/4$, we have that $\cos(2v) < 0$, hence, $\lim_{n \rightarrow \infty} e^{2n\pi \cos(2v)}$ goes to zero, which makes the function go to e^{iv}

$$\int_0^{\pi} \lim_{n \rightarrow \infty} \operatorname{Im} \left(\frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} \right) dv = \int_{\pi/4}^{3\pi/4} \operatorname{Im} \left(\frac{e^{iv}}{0 + 1} \right) dv = \int_{\pi/4}^{3\pi/4} \sin(v) dv = \sqrt{2} \quad (8)$$

The singular points at $v = \pi/4, 3\pi/4$ can be ignored because it makes the limit go to $1/2$, which is a finite singular point.

Applying this to 7:

$$\boxed{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{e^{x^2} + 1} dx = \lim_{n \rightarrow \infty} (2\sqrt{n} - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}})} \quad (9)$$

We set :

$$\boxed{A_n = 2\sqrt{n} - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}}} \quad (10)$$

$$\boxed{T_n = 2\sqrt{n} - \sum_{k=1}^n \frac{1}{\sqrt{k}}} \quad (11)$$

We have :

$$A_n - T_n = 2\sqrt{n} - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} - 2\sqrt{n} + \sum_{k=1}^n \frac{1}{\sqrt{k}} = \sqrt{2} \left(\sum_{k=1}^n \frac{1}{\sqrt{2k}} - \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} \right)$$

$$A_n - T_n = \sqrt{2} \sum_{k=1}^{2n} \frac{(-1)^k}{\sqrt{k}} \longleftrightarrow \boxed{T_n = A_n + \sqrt{2} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}}} \quad (12)$$

We can also have, from 10 11 :

$$A_n + T_n = 4\sqrt{n} - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} - \sqrt{2} \sum_{k=1}^n \frac{1}{\sqrt{2k}} = 4\sqrt{n} - \sqrt{2} \left(\sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} + \sum_{k=1}^n \frac{1}{\sqrt{2k}} \right)$$

$$\boxed{A_n + T_n = \sqrt{2} (2\sqrt{2n} - \sum_{k=1}^{2n} \frac{1}{\sqrt{k}}) = T_{2n} \sqrt{2}} \quad (13)$$

Combining 13 with 12 :

$$A_n + A_n + \sqrt{2} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}} = (A_{2n} + \sqrt{2} \sum_{k=1}^{4n} \frac{(-1)^{k+1}}{\sqrt{k}}) \sqrt{2}$$

$$\boxed{\sqrt{2} A_{2n} - 2A_n = \sqrt{2} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}} - 2 \sum_{k=1}^{4n} \frac{(-1)^{k+1}}{\sqrt{k}}} \quad (14)$$

We can have that, from 10:

$$A_{n+1} - A_n = 2\sqrt{n+1} - 2\sqrt{n} - \sum_{k=0}^n \frac{1}{\sqrt{2k+1}} + \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} = 2\sqrt{n+1} - 2\sqrt{n} - \frac{1}{\sqrt{2n+1}}$$

We see that the function $2\sqrt{n+1} - 2\sqrt{n} - \frac{1}{\sqrt{2n+1}}$ is always positive for all integers including zero n , implying $0 < A_{n+1} - A_n$, $A_n < A_{n+1}$, hence A_n strictly increases.

We know that the limit $\lim_{n \rightarrow \infty} A_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{e^{x^2} + 1} dx$ must converge since the integral is between 0 and $\frac{\pi}{\sqrt{2}}$ 6, implying, because A_n is strictly increasing, $0 \leq A_n < \lim_{n \rightarrow \infty} A_n \leq$

$\sqrt{\pi/2}$. This bounds A_n , satisfying the monotone convergence theorem for A_n . We then set $f_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}}$

$$f_{n+1} - f_n = \sum_{k=1}^{2n+2} \frac{(-1)^{k+1}}{\sqrt{k}} - \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}} = \frac{1}{\sqrt{2n+1}} - \frac{1}{\sqrt{2n+2}}$$

We can also see that $\frac{1}{\sqrt{2n+1}} - \frac{1}{\sqrt{2n+2}}$ is always positive, implying f_n is increasing. Furthermore, the convergence of $\lim_{n \rightarrow \infty} f_n$ is ensured from the alternating series test. We can also have that $\sum_{k=1}^n \frac{(-1)^{k+1}}{\sqrt{k}}$ is at it's highest when $n = 1$, bounding f_n by 1, the monotone convergence theorem is then applicable to f_n and A_n [2]. Hence, setting $\lim_{n \rightarrow \infty} A_n = A$, from 14 :

$$\begin{aligned} \sqrt{2}A - 2A &= \lim_{n \rightarrow \infty} (\sqrt{2}A_{2n} - 2A_n) = \lim_{n \rightarrow \infty} \left(\sqrt{2} \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{\sqrt{k}} - 2 \sum_{k=1}^{4n} \frac{(-1)^{k+1}}{\sqrt{k}} \right) \\ (\sqrt{2} - 2)A &= (\sqrt{2} - 2) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} \end{aligned}$$

$$A = \lim_{n \rightarrow \infty} \left(2\sqrt{n} - \sqrt{2} \sum_{k=0}^{n-1} \frac{1}{\sqrt{2k+1}} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

From 9 :

$$\boxed{\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} = \int_{-\infty}^{\infty} \frac{1}{e^{x^2} + 1} dx} \quad (15)$$

We have that, from finite sums :

$$\sum_{w=1}^n (-1)^{w+1} x^w = \frac{x - (-1)^n x^{n+1}}{1 + x}$$

Setting $x \rightarrow e^{-x^2}$

$$\begin{aligned} \sum_{w=1}^n (-1)^{w+1} e^{-wx^2} &= \frac{e^{-x^2} - (-1)^n e^{-x^2(n+1)}}{1 + e^{-x^2}} = \frac{1 - (-1)^n e^{-x^2n}}{e^{x^2} + 1} \\ \frac{(-1)^n e^{-x^2n}}{e^{x^2} + 1} + \sum_{w=1}^n (-1)^{w+1} e^{-wx^2} &= \frac{1}{e^{x^2} + 1} \end{aligned}$$

Applying this to 15 :

$$\begin{aligned} \sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} &= \int_{-\infty}^{\infty} \frac{(-1)^n e^{-x^2n}}{e^{x^2} + 1} + \sum_{w=1}^n (-1)^{w+1} e^{-wx^2} dx \\ &= \lim_{b \rightarrow \infty} \left(\int_{-b}^b \frac{(-1)^n e^{-x^2n}}{e^{x^2} + 1} dx + \int_{-b}^b \sum_{w=1}^n (-1)^{w+1} e^{-wx^2} dx \right) \end{aligned}$$

Because finite sums interchange with integration :

$$\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} = \lim_{b \rightarrow \infty} \left(\int_{-b}^b \frac{(-1)^n e^{-x^2n}}{e^{x^2} + 1} dx + \sum_{w=1}^n \int_{-b}^b (-1)^{w+1} e^{-wx^2} dx \right)$$

Substituting $x\sqrt{w} = y$, $wx^2 = y^2$, $dx = dy/\sqrt{w}$ on the right integral :

$$\begin{aligned}
\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} &= \lim_{b \rightarrow \infty} \left(\int_{-b}^b \frac{(-1)^n e^{-x^2 n}}{e^{x^2} + 1} dx + \sum_{w=1}^n \int_{-b\sqrt{w}}^{b\sqrt{w}} (-1)^{w+1} e^{-y^2} \frac{dy}{\sqrt{w}} \right) \\
&= \lim_{b \rightarrow \infty} \left(\int_{-b}^b \frac{(-1)^n e^{-x^2 n}}{e^{x^2} + 1} dx + \sum_{w=1}^n \frac{(-1)^{w+1}}{\sqrt{w}} \int_{-b\sqrt{w}}^{b\sqrt{w}} e^{-y^2} dy \right)
\end{aligned}$$

Because $0 \leq e^{-x^2} \leq 1/(x^2 + 1)$ bounds the convergence of $\int_{-\infty}^{\infty} e^{-x^2} dx$ between 0 and π ,

we see that $\lim_{b \rightarrow \infty} \int_{-b\sqrt{w}}^{b\sqrt{w}} e^{-y^2} dy$ converges between 0 and π , hence we can separate the limit :

$$\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} = \int_{-\infty}^{\infty} \frac{(-1)^n e^{-x^2 n}}{e^{x^2} + 1} dx + \sum_{w=1}^{\infty} \frac{(-1)^{w+1}}{\sqrt{w}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

Setting $B_n = (-1)^n \int_{-\infty}^{\infty} \frac{e^{-x^2 n}}{e^{x^2} + 1} dx$, because $\frac{1}{e^{x^2} + 1} \leq \frac{1}{x^2 + 2} \leq \frac{1}{2}$:

$$0 \leq |B_n| \leq \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-x^2 n}}{x^2 + 2} dx \leq \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2 n} dx = \frac{1}{2\sqrt{n}} \int_{-\infty}^{\infty} e^{-x^2} dx$$

We already know $\int_{-\infty}^{\infty} e^{-x^2} dx$ converges to a constant between 0 and π , hence, letting n go to ∞ :

$$0 \leq \lim_{n \rightarrow \infty} |B_n| \leq \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}} \int_{-\infty}^{\infty} e^{-x^2} dx = 0$$

$0 \leq \lim_{n \rightarrow \infty} |B_n| \leq 0$ gives us $0 = \lim_{n \rightarrow \infty} |B_n|$, because limits and absolute value commute, $0 = |\lim_{n \rightarrow \infty} B_n|$, which implies $0 = \lim_{n \rightarrow \infty} B_n$. If we let n go to ∞ :

$$\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} = \lim_{n \rightarrow \infty} \left(B_n + \sum_{w=1}^n \frac{(-1)^{w+1}}{\sqrt{w}} \int_{-\infty}^{\infty} e^{-y^2} dy \right) = \sum_{w=1}^{\infty} \frac{(-1)^{w+1}}{\sqrt{w}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$\sqrt{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} = \sum_{w=1}^{\infty} \frac{(-1)^{w+1}}{\sqrt{w}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-y^2} dy$$

“Cauchy’s theorem cannot be employed to evaluate all definite integrals; thus $\int_0^{\infty} e^{-x^2} dx$ has not been evaluated except by other methods.”, Watson, 1914, page 79, [3]

2 Additional formulas and link to Riemann Zeta

There are a few extra formulas for solving the Gaussian integral that can be derived from this paper.

From 2, if we set H_n such that it is lesser than the absolute value of the first pole, 1, $H_n < \sqrt{\pi}$, we get that there would be no poles inside our contour hence no residue, thus :

$$\int_{-H}^H \frac{1}{e^{x^2} + 1} dx = - \int_0^{\pi} \frac{iH e^{iv}}{e^{H^2 e^{2iv}} + 1} dv \quad (0 \leq H < \sqrt{\pi})$$

With this, we can get that the integral is actually always real as long as H is real, hence:

$$0 = \operatorname{Re}\left(\int_0^{\pi} \frac{e^{iv}}{e^{H e^{2iv}} + 1} dv\right) \quad (H \in \mathbb{R})$$

With this and 8, we can get :

$$i\sqrt{2} = \lim_{n \rightarrow \infty} \int_0^{\pi} \frac{e^{iv}}{e^{2n\pi e^{2iv}} + 1} dv$$

From [5] :

$$\zeta(x) = \lim_{n \rightarrow \infty} \left(\frac{n^{1-x}}{1-x} - \sum_{k=1}^n \frac{1}{k^x} \right) (0 < \operatorname{Re}(x))$$

$$\zeta(1/2) = \lim_{n \rightarrow \infty} \left(2\sqrt{n} - \sum_{k=1}^n \frac{1}{\sqrt{k}} \right)$$

If we take 9 11 and 13 and apply a limit, we would get :

$$\sqrt{\pi} \zeta(1/2) (1 - \sqrt{2}) = \int_{-\infty}^{\infty} \frac{1}{e^{x^2} + 1} dx$$

Furthermore, if we instead take 15 and [4]

$$\zeta(x) = \frac{1}{1 - 2^{1-x}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^x} (0 < \operatorname{Re}(x))$$

$$\zeta(1/2) (1 - \sqrt{2}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

We would also get :

$$\sqrt{\pi} \zeta(1/2) (1 - \sqrt{2}) = \int_{-\infty}^{\infty} \frac{1}{e^{x^2} + 1} dx$$

Which incites the proposition that this paper is actually a proof of the Riemann Zeta functional equation centered at $s = 1/2$. If one were to prove it in a way similar to this paper, one would need to :

Define a function as :

$$f(y) = \int_0^{\infty} \frac{dx}{e^{x^y} + 1}$$

Use finite sums like below and the same approach used before

$$\sum_{w=1}^n (-1)^{w+1} x^w = \frac{x - (-1)^n x^{n+1}}{1+x}$$

To get
$$\frac{1}{y} \sum_{w=1}^{\infty} (-1)^{w+1} w^{-1/y} \int_0^{\infty} e^{-k} k^{1/y-1} dk = \sum_{w=1}^{\infty} (-1)^{w+1} \int_0^{\infty} e^{-wx^y} dx = f(y)$$

By [4] and the definition of the gamma function :

$$\frac{(1 - 2^{1-1/y})}{y} \zeta(1/y) \Gamma(1/y) = f(y)$$

And then, using contour integration, obtaining a formula for $f(y)$ that is similar to 9, doing the same manipulation tricks of 10 and 11.

3 References

References

- [1] Keith Conrad. The gaussian integral. URL: <https://kconrad.math.uconn.edu/blurbs/analysis/gaussianintegral.pdf>.
- [2] none. Monotone convergence theorem, corollary 1. URL: <http://mathonline.wikidot.com/the-monotone-convergence-theorem>.
- [3] G. N. Watson. Complex integration and cauchy's theorem, 1914.
- [4] Wolfram|Alpha. Infinite alternating sum definition of the riemann zeta function for $0 < \text{re}(x)$. URL: <https://functions.wolfram.com/ZetaFunctionsandPolylogarithms/Zeta/06/04/0002/>.
- [5] S. C. Woon. A new representation of the riemann zeta function $\zeta(s)$ (editor's note : the one that was used is at the end of page 5), 1998. [arXiv:math/9812143](https://arxiv.org/abs/math/9812143).