Equilibrium of Charges and Differential Equations Solved by Polynomials II

Igor Loutsenko and Oksana Yermolayeva

Laboratoire de Physique Mathématique, CRM, Université de Montréal

Abstract

We continue study of equilibrium of two species of 2d coulomb charges (or point vortices in 2d ideal fluid) started in [19]. Although for two species of vortices with circulation ratio -1 the relationship between the equilibria and the factorization/Darboux transformation of the Schrodinger operator was established a long ago, the question about similar relationship for the ratio -2 remained unanswered. Here we present the answer.

1 Introduction

In the absence of external field or background flow, equilibrium positions $z_i = x_i + \mathrm{i}y_i$ of N coulomb charges or point vortices on the plane are roots of the system of N algebraic equations

$$\sum_{j=1, j\neq i}^{N} \frac{Q_i}{z_i - z_j} = 0, \quad i = 1, \dots, N, \tag{1}$$

where Q_i is a value of the ith charge, or circulation of the ith vortex (for introduction to theory of vortex equilibria see e.g. [2, 3, 8, 17, 21, 24, 25] and references therein). Without loss of generality, for a system consisting of two distinct species of $N = l + m$ vortices we can choose

$$Q_i = \begin{cases} -1, & i = 1 \dots l \\ \Lambda, & i = l+1 \dots l+m \end{cases}.$$

For such a system the equilibrium condition (1) can be rewritten in terms of bilinear differential equation for two polynomials $p(z)$ and $q(z)$ (of the lth and mth degree correspondingly)

$$p''q - 2\Lambda p'q' + \Lambda^2 pq'' = 0, \tag{2}$$

where prime denotes differentiation wrt z and

$$p(z) = \prod_{i=1}^{l}(z - z_i), \quad q(z) = \prod_{i=1}^{m}(z - z_{l+i}). \tag{3}$$

Since we consider two distinct species, p and q do not have common or multiple roots (for cases with common/multiple roots see e.g. [10] and references therein).

When $\Lambda = 1$ (two species with opposite circulations of equal magnitude), eq. (2) becomes

$$p''q - 2p'q' + pq'' = 0. \tag{4}$$

It first appeared and was completely solved in the paper by Burchnall and Chaundy [6]. Later this case emerged in work by Tkachenko [27] in context of vortex equilibrium ("Tkachenko equation"). Bartmann [7] identified (4) with recurrence relation for the Adler-Moser polynomials $P_n(z)$

$$q(z) = P_n(z), \quad p(z) = P_{n+1}(z), \quad n \in \mathbb{Z}_{\geq 0},$$

which are polynomial τ-functions of the KdV hierarchy [1]. The nth Adler-Moser polynomial is of degree $n(n+1)/2$ in z and also depends non-trivially on additional $n-1$ free parameters s_i, i.e.

$$P_0 = 1, \quad P_1 = z, \quad P_2 = z^3 + s_1, \quad \ldots \quad P_n = P_n(z, s_1, \ldots, s_{n-1}).$$

The Adler-Moser polynomials also satisfy the first-order differential recurrence relation [6, 1]

$$P'_{n+1}P_{n-1} - P'_{n-1}P_{n+1} = (2n+1)P_n^2. \tag{5}$$

The rational solutions of the KdV hierarchy

$$u_n = -2(\log P_n)''$$

are potentials of the Schrodinger operators (in what follows ∂ stands for differentiation wrt z)

$$H_n = -\partial^2 + u_n,$$

that are the "zero energy level" Darboux transforms of the free Schrodinger operator $H_0 = -\partial^2$. Operator H_n is the (rational) Lax operator of the KdV hierarchy.

In his work [7], Bartmann also considered the case $\Lambda = 2$ of (2)

$$p''q - 4p'q' + 4pq'' = 0. \tag{6}$$

This case was further studied in [19, 9, 10, 26, 17]. Equation (6) can be solved completely [19, 21] using an approach similar to that by Burchnall and Chaundy [6]. Similarly to (5), in the $\Lambda = 2$ case, pairs q_n, p_n or q_n, p_{n-1} solving (6) also satisfy the first-order recurrence relation [7, 19]

$$\begin{array}{l} q'_{n+1}q_n - q_{n+1}q'_n = (3n+1)p_n, \\ p'_n p_{n-1} - p_n p'_{n-1} = (6n-1)q_n^4 \end{array}, \quad n \in \mathbb{Z}. \tag{7}$$

However, in difference from the $\Lambda = 1$ case, the procedure for constructing solutions by applications of Darboux transformations remained unknown. Demina and Kudryashov observed that solutions of (6) are related to the Sawada-Kotera hierarchy [9]. This led us to the idea to consider the Darboux transformation for differential operators of the third order, since the Lax operator of the Saswada-Kotera hierarchy has this order [1]. The Darboux transformations for the third-order operators were studied in [5, 23, 4]. To generate solutions of (6) we will introduce "zero level" extension of such transformations.

Before going into detail on the third-order operators, we will briefly recall method of generation of equilibrium configurations through Darboux transformations of the second-order Schrodinger operators (for more detail and review, see e.g. [21]).

[1]Demina and Kudryahov considered different aspect of the relationship, namely connection with self-similar reductions of Sawada- Kotera and Kaupp-Kupershmidt hierarchies studied earlier by Kudryashov [18] . For more detail on Sawada-Kotera and Kaup-Kupershmidt hierarchies, see e.g. [5, 12, 15, 14] and references therein.

2 Darboux Transformations of Second Order Operators and Equilibrium of Charges

The Schrodinger operator

$$H = -\partial^2 + u \tag{8}$$

can be written as

$$H = A^*A + \eta, \tag{9}$$

where A and A^* are the first order differential operators

$$A^* = -\varkappa^{-1}\partial\varkappa = -\partial - \varkappa'/\varkappa, \quad A = \varkappa\partial\varkappa^{-1} = \partial - \varkappa'/\varkappa. \tag{10}$$

Here, function $\varkappa = \varkappa(z)$ is an eigenfunction of H corresponding to the eigenvalue η

$$H\varkappa = \eta\varkappa. \tag{11}$$

Permuting factors A^* and A in (9), we obtain the new Schrodinger operator $\hat{H}$:

$$\hat{H} = AA^* + \eta = -\partial^2 + \hat{u}, \quad \hat{u} = u - 2(\log\varkappa)''. \tag{12}$$

From (9) and (12) it follows that for any λ

$$A(H - \lambda) = (\hat{H} - \lambda)A.$$

The above identity implies that the transformation

$$\hat{\psi} = A\psi = \psi' - \frac{\varkappa'}{\varkappa}\psi \tag{13}$$

maps eigenfunctions corresponding to eigenvalue λ of H to those of $\hat{H}$. Thus, given the "seed" eigenfunction $\varkappa = \psi(z;\eta)$ of H, the new operator $\hat{H}$ and its eigenfunctions $\hat{\psi}(z;\lambda)$ can be obtained from H and its eigenfunctions $\psi(x;\lambda)$. The transformation $H \to \hat{H}$, $\psi \to \hat{\psi}$ given by (12,13) is called the Darboux Transformation.

When $\lambda \neq \eta$ the transform (13) of the two dimensional eigenspace of H is also two-dimensional. However, when $\lambda = \eta$, operator A annihilates eigenfunction $\varkappa$ and (13) maps two dimensional eigenspace of H into one-dimentional sub-eigenspace of $\hat{H}$.

Since, according to (10, 12), A^* annihilates $1/\varkappa$ and $\hat{H} = AA^* + \eta$, we see that $1/\varkappa$ is an eigenfunction of $\hat{H}$ corresponding to the eigenvalue η. Then, the general solution $\hat{\varkappa}$ of $\hat{H}\hat{\varkappa} = \eta\hat{\varkappa}$ is

$$\hat{\varkappa} = \frac{C}{\varkappa}\int \varkappa^2 dz, \tag{14}$$

where C is an arbitrary constant, and the primitive of $\varkappa^2$ includes another arbitrary constant of integration. This extends the Darboux transformation (13) of an eigenfunction to the case $\lambda = \eta$. Transformation (14) (together with (12)) is usually considered for $\eta = 0$ and is called the Darboux transformation at zero energy level.

The Adler-Moser polynomials can be generated by iterative applications of (14). Indeed, rewriting the Tkachenko equation (4) in the Schrodinger form

$$\left(-\partial^2 - 2(\log q)''\right)\left[\frac{p}{q}\right] = 0,$$

we can apply zero-enegy level Darboux transformation (12, 14) to $\varkappa = p/q$ and $u = -2(\log q)''$. In this way we construct the sequence $\varkappa_n = P_{n+1}/P_n$, $u_n = -2(\log P_n)''$ starting from $\varkappa_0 = z$ and $u_0 = 0$ (free Schrodinger operator $H_0 = -\partial^2$). Indeed, rewriting (14) in differential form with

$$\varkappa = P_n/P_{n-1}, \quad \hat{\varkappa} = P_{n+1}/P_n \tag{15}$$

we get the first-order recurrence relation for the Adler-Moser polynomials (5). Therefore, given the initial conditions $P_0 = 1$, $P_1 = z$, the Darboux transformation (14,15) generates all possible equilibrium configurations of charges with $Q_i \in \{-1, 1\}$.

3 $\Lambda = 2$ Case and Third-Order Operators

First, we briefly outline approach by Burchnal and Chaundy [6] adopted to the $\Lambda = 2$ case in [19]: Bilinear equation (6) with $q = q_n$ can be considered as a second-order linear equation for p whose two linearly independent solutions are $p = p_{n-1}$ and $p = p_n$. Analogously, (6) is a linear equation for q, with coefficients defined by $p = p_n$, whose two linearly independent solutions are $q = q_n$ and $q = q_{n+1}$. First-order recurrence relations (7) are the Abel identities relating linearly independent solutions. These identities, together with (6), imply that if the first linearly independent solution is a polynomial without multiple roots, so is the second solution (for detail see e.g. [19, 21]). Therefore, from a polynomial solution q_n, p_{n-1} of (6) we get a polynomial p_n. At the next step we get polynomial q_{n+1} from solution q_n, p_n etc. In this way, a sequence of solutions of (6) can be generated.

Note, that in difference from the $\Lambda = 1$ case (where $n \in \mathbb{Z}_{\geq 0}$), in the $\Lambda = 2$ case the above procedure can be continued indefinitely for both increasing and decreasing n ($n \in \mathbb{Z}$). Pairs q_n, p_{n-1} and q_n, p_n constitute complete set of polynomial solutions of (6) that do not have multiple/common roots [19, 21]. Degrees of q_n and p_n are $n(3n-1)/2$ and $n(3n+2)$ respectively. Examples of several first polynomials for $n \geq 0$ are

$$\begin{array}{ll}
q_0 = 1 & p_0 = 1 \\
q_1 = z & p_1 = z^5 + t_1 \\
 & p_2 = z^{16} + \frac{44}{7} s_2 z^{12} - 32 t_1 z^{11} + 22 {s_2}^2 z^8 - \frac{2112}{7} t_1 s_2 z^7 \\
q_2 = z^5 + s_2 z - 4 t_1 & \qquad + 1408 {t_1}^2 z^6 + t_2 z^5 - 44 {s_2}^3 z^4 + 352 t_1 {s_2}^2 z^3 \\
 & \qquad - 1408 s_2 {t_1}^2 z^2 + 2816 {t_1}^3 z + t_2 t_1 - \frac{11}{5} {s_2}^4 \\
\dots & \dots
\end{array} \tag{16}$$

etc, while for $n \leq 0$

$$\begin{array}{ll}
q_0 = 1 & p_0 = 1 \\
q_{-1} = z^2 + s_{-1} & p_{-1} = z \\
q_{-2} = z^7 + 7 s_{-1} z^5 + 35 s_{-1}^2 z^3 + s_{-2} z^2 & p_{-2} = z^8 + \dfrac{28}{5} s_{-1} z^6 + 14 s_{-1}^2 z^4 \\
\qquad - 35 s_{-1}^3 z + s_{-1} s_{-2} - \dfrac{5}{2} t_{-2} & \qquad + 28 s_{-1}^3 z^2 + t_{-2} z - 7 s_{-1}^4 \\
\dots & \dots
\end{array} \tag{17}$$

etc, where t_i and s_i stand for arbitrary complex constants [2] emerging in course of integrations of Abel identities.

[2]We set constant corresponding to shift of z to zero, so, without lost of generality, $q_1 = p_{-1} = z$

Now we are ready to look for Darboux transformation in $\Lambda = 2$ case: Similarly to the $\Lambda = 1$ case, the bilinear equation (6), corresponding to $\Lambda = 2$ case can be rewritten in the Schrodinger form

$$\left(-\partial^2 - 6(\log q)''\right)\left[\frac{p}{q^2}\right] = 0, \tag{18}$$

i.e. for sequences of p_n and q_n we have

$$\left(-\partial^2 + u_n\right)\phi = 0, \quad u_n = -6(\log q_n)'', \tag{19}$$

where

$$\phi = p_n/q_n^2 \quad \text{or} \quad \phi = p_{n-1}/q_n^2. \tag{20}$$

However, we cannot apply Darboux transformations to the above second order-operators, because now $u_n = -6(\log q_n)''$ (cf. (12)). To proceed, we note that according to the Abel identities (7), p_n in (19, 20) can be replaced with $q'_{n+1}q_n - q_{n+1}q'_n$ and we get the third-order differential equation

$$L\varkappa = 0, \quad L = \partial^3 - u_n\partial \tag{21}$$

for $\varkappa$, where $\varkappa$ is any linear combination of q_{n+1}/q_n, q_{n-1}/q_n and 1. Because q_{n-1}, q_n, q_{n+1} have distinct degrees, this combination is the general solution of (21). Since the linear combination $C_1q_{n+1} + C_0q_n + C_{-1}q_{n-1}$ is a polynomial, any solution of (21) corresponds to an equilibrium configuration. Polynomial $C_1q_{n+1} + C_0q_n + C_{-1}q_{n-1}$ is a re-parametrized q_{n+1} for $n > 0$ or re-parametrized q_{n-1} for $n \le 0$ respectively.

Thus, $\varkappa = q_{n\pm1}/q_n$, where $n \ge 0$ or $n \le 0$ respectively, are general solutions of (21). They correspond to complete set of equilibrium configurations of two species of charges with $Q_i \in \{-1, 2\}$.

4 Darboux Transformations for Third-Order Operators

Darboux transformations for the third-order operators of the type (21), i.e. transformations $L \to \hat{L}$, $\psi \to \hat{\psi}$ with the seed function $\varkappa$, where

$$L\psi = \lambda\psi, \quad L\varkappa = \eta\varkappa, \quad L = \partial^3 - u\partial, \quad \hat{L}\hat{\psi} = \lambda\hat{\psi}, \quad \hat{L} = \partial^3 - \hat{u}\partial, \tag{22}$$

were found by Aiyer et al [5]. Their derivation, via factorization, was presented by Athorne and Nimmo in [4]. Similarly to the $\Lambda = 1$ case, we need their "zero-level" extensions (for $\lambda = \eta = 0$) which we will find in this section.

Following [4], we first review the case of $\eta = 0$ and arbitrary λ. Operator L can be presented through the product of the second and the first-order factors

$$L = BA, \tag{23}$$

such that

$$B = \partial^2 + f\partial - f' - \frac{f''}{f}, \quad A = \partial - f, \quad f = \frac{\varkappa'}{\varkappa}. \tag{24}$$

In contrast to the case of the Schrodinger operator, permutation of factors in (23) maps L into $\hat{L}$ of the similar type (i.e. as in (22)) just in the trivial case $u = \text{const}$. Non-trivial result arises from the transformation

$$A\psi = \hat{A}\hat{\psi}, \quad \hat{A} = \partial - \hat{f}. \tag{25}$$

Multiplying equation $L\psi = \lambda\psi$ by A on the left and taking (23) and (25) into account we get

$$AB\hat{A}\hat{\psi} = \lambda\hat{A}\hat{\psi}.$$

Demanding that

$$AB = \hat{A}\hat{B}, \quad \hat{B} = \partial^2 + \hat{f}\partial - \hat{f}' - \frac{\hat{f}''}{\hat{f}} \tag{26}$$

we see that, up to element of kernel of $\hat{A}$,

$$\hat{B}\hat{A}\hat{\psi} = \lambda\hat{\psi}.$$

Equation (26) has a solution $\hat{f} = -f$ and the Darboux-type transform $\hat{L} = \hat{B}\hat{A}$ is obtained from $L = BA$ by the involution[3] $f \to -f$. From (23, 24) it follows that for such a transformation

$$\hat{u} = u - 6(\log\varkappa)''. \tag{27}$$

Since $\hat{A} = \partial + f = \partial + \varkappa'/\varkappa$, function $\varkappa^{-1}$ is an eigenfunction of $\hat{L} = \hat{B}\hat{A}$ corresponding to $\lambda = 0$:

$$\hat{L}\varkappa^{-1} = 0.$$

Similarly to the Schrodinger case, the complete zero-level transform $\hat{\varkappa}$ of $\varkappa$ is a general solution of equation

$$\hat{L}\hat{\varkappa} = 0. \tag{28}$$

It can be obtained from its particular solution $\varkappa^{-1}$ by elementary methods: Fist, we rewrite (28) as

$$\hat{\phi}'' - \hat{u}\hat{\phi} = 0, \quad \hat{\phi} = \hat{\varkappa}'.$$

One of the linearly independent solution of the above second order equation equals $(\varkappa^{-1})'$. Then, its general solution is

$$\hat{\phi} = C(\varkappa^{-1})' \int \frac{dz}{((\varkappa^{-1})')^2}, \tag{29}$$

where C is an arbitrary constant and the primitive includes an arbitrary constant of integration. To get $\hat{\varkappa}$, we have to integrate (29), i.e. $\hat{\varkappa} = \int \hat{\phi} dz = C \int \left(\int \frac{dz}{((\varkappa^{-1})')^2} \right) d\frac{1}{\varkappa}$. Integrating by parts, we obtain, modulo multiplication by a constant,

$$\hat{\varkappa} = \int \frac{\varkappa^3}{(\varkappa')^2} dz - \frac{1}{\varkappa} \int \frac{\varkappa^4}{(\varkappa')^2} dz, \tag{30}$$

where the primitives include arbitrary constants of integration. Thus, we have found the zero-level Darboux transformations (27, 30) for the third-order operators of the type $\partial^3 - u\partial$.

[3] This is similar to the Schrodinger operator case where the Darboux transformation is the involution $A = \partial - f \leftrightarrow A^* = -(\partial + f)$, cf (10).

5 $\Lambda = 2$ Configurations and Darboux Transformations

As was shown in section 3, the general solution[4] of equation $L\varkappa = 0$, where $L = \partial^3 + 6(\log q_{n-1})''\partial$, is $\varkappa = q_n/q_{n-1}$. According to (27), the Darboux transform of L equals $\hat{L} = \partial^3 + 6(\log q_n)''\partial$. Since the transform $\hat{\varkappa}$ of $\varkappa$ is the general solution of $\hat{L}\hat{\varkappa} = 0$, we conclude that $\hat{\varkappa} = q_{n+1}/q_n$ and the Darboux transformations (30) generate sequence q_n.

Note that the more general fact holds: The first-order recurrence relations (7) are, in fact, the Darboux transformations (30) rewritten in differential form. Derivation of these relations doesn't need to involve bilinear equation (6).

Indeed, let us write

$$\hat{\varkappa} = q_+/q, \quad \varkappa = q/q_-, \tag{31}$$

where q_+, q and q_- are some functions of z. Then

$$\hat{\varkappa}' = p/q^2, \quad \varkappa' = p_-/q_-^2, \tag{32}$$

where

$$p = q_+'q - q_+q', \quad p_- = q'q_- - qq_-'. \tag{33}$$

Substituting (31,32) into transformation (30), or equivalently, substituting $\hat{\phi} = p/q^2$, $(\varkappa^{-1})' = -\varkappa^{-2}p_-/q_-^2$ and $\varkappa = q/q_-$ into (29) and rewriting result in the differential form, we obtain

$$q^4 \propto p'p_- - pp_-'. \tag{34}$$

Equations (33, 34) are, modulo constant factors, the first-order recurrence relations (7) with $q_+ = q_{n+1}$, $q = q_n$, $q_- = q_{n-1}$ and $p = p_n$, $p_- = p_{n-1}$. Together with initial conditions $q_0 = p_0 = 1$, they produce solutions of (6). There also exist other initial conditions for which solutions of (7) are polynomials that are not solutions of (6). They will be considered in the next section.

In summary: All polynomial solutions of the bilinear equation (6) that do not have common/multiple roots can be generated by iterations of transformation (30) with

$$\hat{\varkappa} = q_{n+1}/q_n, \quad \varkappa = q_n/q_{n-1} \tag{35}$$

together with equation (cf. (33))

$$p_n \propto q_{n+1}'q_n - q_{n+1}q_n' \tag{36}$$

and initial conditions $q_0 = p_0 = 1$. Thus, starting from $\varkappa = \varkappa_0 = z$, iterations of transformation (30) produce rational functions $\varkappa_n = q_{n+1}/q_n$, such that zeros of $\varkappa_n'$ correspond to positions of charges with $Q_i = -1$, while poles of $\varkappa_n'$ correspond to positions of charges with $Q_i = 2$.

6 Darboux Transformations and Terminating Configurations

Introducing factorizable function

$$\phi = \text{const} \prod_{i=1}^{N} (z - z_i)^{Q_i}, \tag{37}$$

[4] Here, for simplicity, we consider situation with increasing $n \geq 0$.

one can easily show that the equilibrium conditions (1) are equivalent to the absence of simple poles in the potential u of the Schrodinger equation $(-\partial^2 + u)\phi = 0$.

In the case of systems related to the second-order operators, from (12) it follows that the transform $\hat{u}$ of u, where $\varkappa = \phi$, do not have simple poles. Therefore, provided the transform $\hat{\varkappa}$ of $\varkappa$ is also factorizable, it corresponds to an equilibrium configuration.

Similar fact takes place in the third-order case. Indeed, due to (27), simple poles are absent in the potential of the Schrodinger equation $(-\partial^2 + \hat{u})\hat{\phi} = 0$, $\hat{\phi} = \hat{\varkappa}'$. Therefore, Darboux transformation maps equilibrium configuration into another equilibrium configuration in both second and third-order cases, provided the transform $\hat{\varkappa}$ of $\varkappa$ is factorizable.

The set of equilibrium configurations that can be generated by Darboux transformations is bigger than the set of solutions (4) or (6) and, apart from configurations of two species of charges, also comprises configurations with three species: Factorizable $\varkappa$ need not to necessarily correspond to potentials $u = -2(\log q)''$ in the second-order or $u = -6(\log q)''$ in the third order case.

As was mentioned in section 2, the first-order recurrence relation (5) for P_n follows from the Darboux transformation (14, 15). Bilinear equation (4) is not involved in this derivation. Therefore, to get a sequence of equilibrium configurations, the initial conditions for (5) need not necessarily to be those for the Adler-Moser polynomials $P_0 = 1$, $P_1 = z$. A family of sequences with other initial conditions was introduced in work by Duistermmat and Grunbaum [11] in the context of bi-spectral problem. In relation to vortex equilibria this family appeared in work by O'Neil and Cox-Steib [26] (also see [17]). There, sequences start from $P_0 = 1$, $P_1 = z^{1/2}$. In difference from the Adler-Moser case, these sequences terminate as a non-factorizable $\varkappa$ emerges at some step of recurrence. Examples of first several sequences are:

$$\begin{aligned}
&P_0 = 1, \quad P_1 = z^{1/2}, \quad P_2 = z^2 + t_1.\\
&P_0 = 1, \quad P_1 = z^{1/2}, \quad P_2 = z^2, \quad P_3 = z^{9/2} + t_2 z^{1/2}, \quad P_4 = z^8 + 6t_2 z^4 + t_3 z^2 - 3t_2^2.\\
&\dots
\end{aligned} \tag{38}$$

etc. Here, a half-integer charge of the third specie is present at the origin $z = 0$.

Similar situation occurs in the case of the third-order operators. As we saw in section 5, the Darboux transformations (30, 35), together with (36), result in the first-order recurrence relations (7). For two species of charges initial conditions are $p_0 = q_0 = 1$. However, there exist other initial conditions, where terminating sequences of equilibrium configurations, that are not solutions of (6), are generated by the Darboux transformations. Such sequences of solutions were found in [26] (also see [9], [17]). For example, for $q_0 = 1$, $p_0 = z^2$ the first several sequences are

$$\begin{aligned}
&q_0 = 1, \quad p_0 = z^2, \quad q_1 = z^3 + t_1.\\
&q_0 = 1, \quad p_0 = z^2, \quad q_1 = z^3, \quad p_1 = z^2(z^9 + t_2), \quad q_2 = z^9 + t_3 z^3 - 2t_2.\\
&\dots
\end{aligned}$$

and so on.

7 Conclusions and Open Problems

In this article we found Darboux-type transformations that generate complete set of equilibrium configuration of two species of vortices with circulation ratio -2. They also generate sets of configurations, where an additional charge of the third specie is present.

In conclusion, we would like to mention open problems and outline several generalizations of systems studied in the present article.

First, we would like to mention problem of determinant representation of functions corresponding to the equilibrium configuration. Wronskian representation of the Adler-Moser polynomials is known since work by Burchnall and Chaundy [6]. This representation can be considered as a zero-level degeneration of the Crum theorem for sequences of the Darboux transformations (for details see e.g. [22, 21]). Using such an approach (see e.g. [21]) one can also derive Wronskian representation for terminating configurations of type (38).

In the third-order case, one has to deal with Pfaffians, rather than Wronskians [23, 15]. "Pfaffian analog" of the Crum theorem exists for a sequence of Darboux transformations associated with the set of distinct eigenvalues [23]. However, it is unclear how to deal with its zero-level degeneration, because, in difference from the second-order case, the eigenfunctions must satisfy certain boundary conditions [5, 23, 4].

Recently, a Pfaffian representation of polynomial τ-functions of the Sawada-Kotera hierarchy was found by Kac and Van de Leur in [15] via free-fermion approach.

It is known from the theory of integrable hierarches that, for a rational Lax operator L, there exists an eigenfunction $\varkappa = \theta/\tau$ corresponding to zero eigenvalue, where τ is a polynomial τ-function and θ is also a polynomial. In the case of the Sawada-Kotera hierarchy $L = \partial^3 - u\partial$, where $u = -6(\log\tau)''$ is a solution of the hierarchy (see e.g. [15] and references therein). Equation $L\varkappa = 0$, rewritten in the Schrodinger form, is equivalent to the bilinear equation (6) for $q = \tau$ and $p = \tau'\theta - \tau\theta'$. Then, since p and q are polynomials, polynomial τ-functions of the Sawada-Kotera hierarchy correspond to equilibrium configurations.

Therefore, the result by Kac and Van de Leur provides a pfaffian representation for equilibrium configurations of two species of charges with $Q_i \in \{-1, 2\}$.

It was shown by Fordy and Gibbons [12] that solutions u of the Sawada-Kotera and solutions v of the Kaup-Kupershmidt hierarchy are related by the Miura transormation

$$u = w' + w^2, \quad v = w' - \frac{1}{2}w^2.$$

It easy to see that for rational solutions of the Sawada-Kotera hierarchy $w = \phi'/\phi$, $\phi = p/q^2$. Then, taking (6) into account, we conclude that the Miura transform of u equals $v = 3(\log p)''$. The last equation is nothing but formula for τ-function of the Kaup-Kupershmidt hierarchy, where $\tau = p$. Therefore, polynomial τ-functions of the Kaup-Kupershmidt hierarchy correspond to $\Lambda = 2$ equilibrium configurations.

Kac and Van de Leur found a determinant representation for polynomial τ-functions of the Kaup-Kupershmidt hierarchy [14]. Thus, apart from the pfaffian, $\Lambda = 2$ equilibrium configurations also have determinant representation.

In this light, it would be interesting to explore possibility of existence of pfaffian or determinant representation for terminating configurations presented in section 6 for the third-order case.

Finally, we would like to outline several generalizations of systems considered in this paper: One of generalizations extends (1) to the case of translating configurations or equilibria of charges in the homogeneous electric field k:

$$k + \sum_{j=1, j\neq i}^{N} \frac{Q_j}{z_i - z_j} = 0, \quad i = 1 \dots N.$$

Extension of the bilinear equation (2) for this case, rewritten in the Schrodinger form, is

$$\left(\partial^2 + \Lambda(\Lambda+1)(\log q)''\right) \Phi = k^2 \Phi, \quad \Phi = \frac{p(z,k)}{q(z,k)^{\Lambda}} e^{kz}.$$

When $\Lambda = 1$, $\Phi(z,k)$ is the "rational" Baker-Akhiezer function of the KdV hierarchy. It can be obtained by application of sequence of the $\eta = 0$ Darboux transformations (13, 14) to the eigenfunction $\psi = e^{kz}$ of the free Schrodinger operator corresponding to the eigenvalue $\lambda = k^2$ (for review see e.g. [21] and references therein).

In contrast to the $\Lambda = 1$ case, the Darboux transformation method does not work when $\Lambda = 2$ and $k \neq 0$. This is because the $\eta = 0$ Darboux transformations of e^{kz} are now associated with another "rational" Baker-Akhiezer function $\Psi(z,k)$, a such that

$$L\Psi = k^3 \Psi,$$

where $L = (\partial^2 + 6(\log q)'')\,\partial$ is of the third order. Only when $k = 0$, the above eigenvalue problem reduces to solution of the Schrodinger equation (18), which is equivalent of (6). Therefore, here Darboux method works only for non-translating (i.e. $k = 0$) configurations. Thus, question of classification of translating configurations remains open for $\Lambda = 2$.

Further generalization deals with the periodic configurations ("vortex streets") satisfying

$$k + \sum_{j=1, j\neq i}^{N} Q_j \cot(z_i - z_j) = 0, \quad i = 1 \dots N.$$

When $\Lambda = 1$, this generalization is associated with the soliton solutions of the KdV hierarchy and "trigonometric" Baker-Akhieser function $\Phi = q(z,k)/p(z,k)e^{kz}$, where now q and p are trigonometric polynomials in z ([16, 20, 13], for review see e.g. [21]). Trigonometric Baker-Akhiezer function is obtained from the eigenfunction e^{kz} (with eigenvalue $\lambda = k^2$) of the free Schrodinger operator by iterations of the Darboux transformations: First with $\eta = k_1^2$, then transformation with $\eta = k_2^2$, etc. Here k_i are integers, such that $0 < k_1 < k_2 < \dots$.

When $\Lambda = 2$, vortex streets configurations cannot be generated by Darboux transformations, even in the non-translating case $k = 0$ only short terminating sequences can be produced by the zero-level transformations. Indeed, similarly to the second order case, operators L with periodic coefficients can be obtained by sequence of Darboux transformations from ∂^3 only when transformations are associated with a sequence of distinct, nonzero η. Here equations

$$L_i \varkappa_i = \eta_i \varkappa_i, \quad L_i = \partial^3 - u_i \partial, \quad \eta_i = k_i^3,$$

where u_i is periodic and $k_i \neq 0$, cannot be reduced to the Schrodinger equations. As a consequence, $\varkappa_i$, do not correspond to equilibrium configurations . It would be interesting to explore the $\Lambda = 2$ vortex street configurations using alternative methods.

References

[1] Adler M., Moser J., *On a class of Polynomials connected with the Korteweg-de-Vries Equation*, Comm. Math. Phys. 1978. V. 61. P. 1-30.

[2] Aref H., *Point vortex dynamics: a classical mathematics playground.* J. Math. Phys. 48, (2007) 065401. (doi:10.1063/1.2425103)

[3] Aref H., *Vortices and polynomials*, Fluid Dyn. Res. 39, 5–23. (2007) (doi:10.1016/ j.fluiddyn.2006.04.004)

[4] C. Athorne and J. J. C. Nimmo, *Darboux theorems and factorization of second- and third-order ordinary differential operators*, Inverse Problems 7 (1991) 645654

[5] Raju N Aiyert, Benno Fuchssteiner and Walter Oevel, *Solitons and discrete eigenfunctions of the recursion operator of non-linear evolution equations: I. The Caudrey-Dodd-Gibbon-Sawada-Kotera equation*, J. Phys. A: Math. Gen. 19 (1986) 3755-3770

[6] Burchnall J.L., Chaundy T.W., *A set of differential equations which can be solved by polynomials*, Proc. Lonlon. Math. Soc. 1929-1930. V. 30. P. 401-414.

[7] Bartman A.B., *A new interpretation of the Adler-Moser KdV polynomials: interaction of vortices*, In Nonlinear and turbulent processes in physics, vol. 3 (ed. RZ Sagdeev), (1984) pp. 1175–1181. New York, NY: Harwood Academic.

[8] Clarkson PA. *Vortices and polynomials*, Stud. Appl. Math. 123, 37–62. (2009) (doi:10.1111/j.1467-9590.2009.00446.x)

[9] Demina, M. and Kudryashov, N., *Point Vortices and Polynomials of the Sawada – Kotera and Kaup – Kupershmidt Equations*, Regul. Chaotic Dyn., 2011, vol. 16, no. 6, pp. 562–576.

[10] Demina, M. and Kudryashov, N., *Vortices and Polynomials: Non-Uniqueness of the Adler –Moser Polynomials for the Tkachenko Equation*, J. Phys. A, 2012, vol. 45, no. 19, 195205, 12 pp.

[11] J.J. Duistermaat and F.A. Grünbaum, *Differential equations in the spectral parameters*, Commun. Math. Phys. **103** (1986), 177–240.

[12] Allan P. Fordy and John Gibbons, *Some Remarkable Nonlinear Transformations*, Phys.Let. A, Volume 75A, number 5, 1980

[13] Adrian D. Hemery and Alexander P. Veselov, *Periodic Vortex Streets and Complex Monodromy*, SIGMA 10 (2014), 114

[14] Viktor Kac and Johan Van de Leur, *The generalized Giambelli formula and polynomial KP and CKP tau-functions*, J. Phys. A: Math. Theor. 56 185203, 2023

[15] Viktor Kac and Johan Van de Leur, *Polynomial Tau-Functions of the n-th Sawada–Kotera Hierarchy*, Mathematics 2024, 12(5), 681

[16] J. Kadtke, L. Campbell, *Method for finding stationary states of point vortices*, Phys. Rev. A 36 (9) (1987) 4360–4370.

[17] Krishnamurthy Vikas S., Wheeler Miles H., Crowdy Darren G. and Constantin Adrian, *A transformation between stationary point vortex equilibria*, Proc. R. Soc. A. (2020) 4762020031020200310

[18] Kudryashov N.A., *Two hierarchies of ordinary differential equations and their properties*, Physics Letters A, 252 (1999) 173 -179

[19] Igor Loutsenko, *Equilibrium of charges and differential equations solved by polynomials*, 2004 J. Phys. A: Math. Gen. 37 1309

[20] Loutsenko, I., *Integrable Dynamics of Charges Related to the Bilinear Hypergeometric Equation*, Commun. Math. Phys. 242, 251–275 (2003). https://doi.org/10.1007/s00220-003-0944-z

[21] Igor Loutsenko, Oksana Yermolayeva, *Vortices and Factorization*, https://arxiv.org/abs/2403.07537

[22] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, 1991

[23] J. J. C. Nimmo, *The Crum Transformation for a Third-Order Scattering Problem*, Proc. R. Soc. Lond. A 1990 431, 361-369

[24] O'Neil KA., *Stationary configurations of point vortices*, Trans. Am.Math. Soc. 302, 383–425 (1987). (doi:10.2307/2000850)

[25] O'Neil KA. , *Minimal polynomial systems for point vortex equilibria*, Phys. D 219, 69–79 (2006). (doi:10.1016/j.physd.2006.05.012)

[26] O'Neil, K.A., Cox-Steib, N., *Generalized Adler-Moser and Loutsenko polynomials for point vortex equilibria*, Regul. Chaot. Dyn. 19, 523–532 (2014), https://doi.org/10.1134/S1560354714050013

[27] Tkachenko V.K., *Studying the Violations of the Superfluidity of Helium in Broad Capillaries under the Influence of Heat Flow*, 1964 Unpublished thesis, Institute of Physical Problems, Moscow